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Dirac Field of Kaluza–Klein Theory in Weitzenböck Space

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The Dirac equation in five-dimensional Weitzenböck space is dervied. The effect of spin–spin interaction induced by torsion is revealed by use of the Dirac equation in the weak-field situation. A comparison is made of the Dirac equation of Kaluza–Klein theory in three types of spaces. It is concluded that, from the point of view of simplicity, the Weitzenböck space is the most suitable one for establishing Kaluza–Klein theory.

1. INTRODUCTION

The Kaluza–Klein (KK) theory (Kaluza, 1921; Klein, 1926; Toms, 1984), which unifies gravity and electromagnetic interaction, has been generalized to non-Riemann space (Macias and Dehnen, 1991; Zhang and Wu, 1996; Wu, 1993; Lee and Wu, 1992). The Dirac equation has been given in five-dimensional (5D) Riemann space and Riemann–Cartan space (Macias and Dehnen, 1991; Zhang and Wu, 1996). In this paper, we derive the Dirac equation in 5D Weitzenböck space. By applying it to the weak-field situation, we see the nature of the spin–spin interaction induced by torsion in this case. Then we compare the form and content of the Dirac equation in 5D KK theory for the three kinds of space, Riemann space (V_5), Weitzenböck space (A_5), and Riemann–Cartan space (U_5).

We use the following conventions: Objects or quantities with (without) a carat ($\wedge$) refer to higher dimensional space (4D space-time) and objects or

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quantities with a tilde ($\sim$) refer to Riemann space; Greek letters μ, ν, ... (Latin letters A, B, ... ; i, j, ...) are used for coordinate basis indices (horizental lift basis indices; indices for orthonormal parallel vector fields).

2. GEOMETRY OF 5D WEITZENBÖCK SPACE

Riemann–Cartan space has both curvature and torsion. If the torsion tensor vanishes, we get Riemann space; setting the curvature tensor to vanish identically, we obtain Weitzenböck space. Because it is curvature-free, the Weitzenböck space has an absolute parallelism, i.e., in 5D Weitzenböck space (A_5), there exists a set of linearly independent, orthonormal, 5-parallel vector fields $\hat{b} = \{\hat{b}_{\hat{k}}{}^{\hat{\mu}}\}$ which satisfy (Lee and Wu, 1992)

$$\hat{b}_{\hat{i}}{}^{\hat{\mu}}\hat{b}^{\hat{i}}{}_{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}}, \qquad \hat{b}_{\hat{i}}{}^{\hat{\mu}}b^{\hat{j}}{}_{\hat{\mu}} = \delta^{\hat{j}}_{\hat{i}} \tag{2.1}$$

$$\hat{b}_{\hat{i}}{}^{\hat{\mu}}\hat{g}_{\hat{\mu}\hat{\nu}}\hat{b}_{\hat{j}}{}^{\hat{\nu}} = \hat{\eta}_{\hat{i}\hat{j}}, \qquad \hat{b}^{\hat{i}}{}_{\hat{\mu}}\hat{\eta}_{\hat{i}\hat{j}}b^{\hat{j}}{}_{\hat{\nu}} = \hat{g}_{\hat{\mu}\hat{\nu}} \tag{2.2}$$

where $\{\hat{b}^{\hat{k}}{}_{\hat{\mu}}\}$ are the dual vector fields of $\{\hat{b}_{\hat{k}}{}^{\hat{\mu}}\}$,$\hat{g}_{\hat{\mu}\hat{\nu}}$ denotes the metric of 5D, space and the Lorentz metric $\hat{\eta}_{\hat{i}\hat{j}} = \mathrm{diag}\,(+ - - - -)$.

The 5D metric components in coordinate basis are given generally by

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} - \kappa^2 A_\mu A_\nu & -\kappa A_\mu \\ -\kappa A_\nu & -1 \end{pmatrix}, \qquad \hat{\mu} = (\mu, 5) \tag{2.3}$$

where $g_{\mu\nu}$ is the 4D space-time metric, $A_\mu(x)$ is the gauge potential of the electromagnetic field, and κ is a constant which makes κA_μ to be dimensionless.

To simplify the calculation, we adopt the horizental lift basis (HLB) $\hat{e}_{\hat{A}}$; its components are given by

$$\hat{e}_A = (\partial_\mu - \kappa A_\mu \partial_5)\delta^\mu_A, \qquad \hat{e}_5 = \partial_5 \tag{2.4}$$

The commutation coefficients of this basis $\hat{C}^{\hat{E}}{}_{\hat{A}\hat{B}}$ can be defined by

$$[\hat{e}_{\hat{A}}, \hat{e}_{\hat{B}}] = \hat{C}^{\hat{E}}{}_{\hat{A}\hat{B}}\, \hat{e}_{\hat{E}} \tag{2.5}$$

A simple calculation gives

$$[\hat{e}_A, \hat{e}_B] = -\kappa F_{\mu\nu}\partial_5 \delta^\mu_A \delta^\nu_B, \qquad [\hat{e}_A, \hat{e}_5] = 0 \tag{2.6}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength of the electromagnetic gauge field. Obviously, the only nonvanishing commutation coefficient is

$$C^5{}_{AB} = -\kappa F_{\mu\nu}\delta^\mu_A \delta^\nu_B \tag{2.7}$$

In the HLB, the metric becomes block diagonal,

$$\hat{g}_{\hat{A}\hat{B}} = \begin{pmatrix} g_{AB} & 0 \\ 0 & -1 \end{pmatrix} \tag{2.8}$$

where $g_{AB} = g_{\mu\nu}\delta^{\mu}_{A}\delta^{\nu}_{B}$. The parallel vector field $\hat{b}_{\hat{i}}{}^{\hat{A}}$ can also be chosen as block diagonal,

$$\hat{b}_{i}{}^{\hat{A}} = \begin{pmatrix} b_{i}{}^{\hat{A}} & 0 \\ 0 & b_{5}{}^{5} \end{pmatrix} \tag{2.9}$$

i.e.,

$$\hat{b}_{i}{}^{A} = b_{i}{}^{A}, \qquad \hat{b}_{i}{}^{5} = \hat{b}_{5}{}^{A} = 0, \qquad \hat{b}_{5}{}^{5} = 1 \tag{2.10}$$

and

$$\hat{b}^{i}{}_{A} = b^{i}{}_{A}, \qquad \hat{b}^{i}{}_{5} = \hat{b}^{5}{}_{A} = 0, \qquad \hat{b}^{5}{}_{5} = 1 \tag{2.11}$$

The 5D affine connection $\hat{\Gamma}^{\hat{E}}{}_{\hat{A}\hat{B}}$ can be defined by (Wu, 1993)

$$\hat{\nabla}_{\hat{e}_5}\, \hat{e}_{\hat{A}} = \hat{\Gamma}^{\hat{E}}{}_{\hat{A}\hat{B}}\, \hat{e}_{\hat{E}} \tag{2.12}$$

Owing to absolute parallelism, the covariant derivative of parallel vector fields $\hat{\nabla}\hat{b} = 0$; then we have

$$\hat{\Gamma}^{\hat{E}}{}_{\hat{A}\hat{B}} = \hat{b}_{\hat{i}}{}^{\hat{E}}\, \hat{e}_{\hat{B}}\hat{b}^{\hat{i}}{}_{\hat{A}} \tag{2.13}$$

Using (2.10) and (2.11), we can evaluate the affine connection and get the only nonvanishing component

$$\hat{\Gamma}^{E}{}_{AB} = \Gamma^{E}{}_{AB} = \Gamma^{\lambda}{}_{\mu\nu}\delta^{E}_{\lambda}\delta^{\mu}_{A}\delta^{\nu}_{B} \tag{2.14}$$

where $\Gamma^{\lambda}{}_{\mu\nu}$ is the 4D connection in coordinate basis. The torsion tensor in HLB can be written as

$$\hat{Q}^{\hat{E}}{}_{\hat{A}\hat{B}} = \hat{\Gamma}^{\hat{E}}{}_{\hat{A}\hat{B}} - \hat{\Gamma}^{\hat{E}}{}_{\hat{B}\hat{A}} + C^{\hat{E}}{}_{\hat{A}\hat{B}} \tag{2.15}$$

It is found by use of (2.14) that

$$\hat{Q}^{\hat{E}}{}_{AB} = Q^{E}{}_{AB}, \qquad \hat{Q}^{5}{}_{AB} = -\kappa F_{AB} \tag{2.16}$$

In order to compare with the cases of Riemann space and Riemann–Cartan space, we decompose the affine connection $\hat{\Gamma}^{\hat{E}}{}_{\hat{A}\hat{B}}$ into a Riemannian and a contortion part,

$$\hat{\Gamma}^{\hat{E}}{}_{\hat{A}\hat{B}} = \hat{\bar{\Gamma}}^{\hat{E}}{}_{\hat{A}\hat{B}} + \hat{K}^{\hat{E}}{}_{\hat{A}\hat{B}} \tag{2.17}$$

where

$$\hat{\bar{\Gamma}}_{\hat{E}\hat{A}\hat{B}} = \tfrac{1}{2}[\hat{e}_{\hat{A}}(\hat{g}_{\hat{E}\hat{B}}) + \hat{e}_{\hat{B}}\,(\hat{g}_{\hat{E}\hat{A}}) - \hat{e}_{\hat{E}}(\hat{g}_{\hat{A}\hat{B}}) + \hat{C}_{\hat{E}\hat{B}\hat{A}} + \hat{C}_{\hat{A}\hat{E}\hat{B}} + \hat{C}_{\hat{B}\hat{E}\hat{A}}] \tag{2.18}$$

represents the Riemann connection in the HLB, and

$$\hat{K}^{\hat{E}}{}_{\hat{A}\hat{B}} = \frac{1}{2}(\hat{Q}^{\hat{E}}{}_{\hat{A}\hat{B}} - \hat{Q}_{\hat{B}}{}^{\hat{E}}{}_{\hat{A}} - \hat{Q}_{\hat{A}}{}^{\hat{E}}{}_{\hat{B}}) \tag{2.19}$$

denotes the 5D contortion tensor. We know from (2.16) that

$$\hat{K}^{E}{}_{AB} = K^{E}{}_{AB} = K^{\lambda}{}_{\mu\nu}\delta^{E}_{\lambda}\delta^{\mu}_{A}\delta^{\nu}_{B}, \qquad \hat{K}^{5}{}_{AB} = -\frac{1}{2}\kappa F_{AB} = -\frac{1}{2}\kappa F_{\mu\nu}\delta^{\mu}_{A}\delta^{\nu}_{B} \tag{2.20}$$

where $\hat{K}^{\lambda}{}_{\mu\nu}$ is the 4D contortion tensor in coordinate basis.

For the same reason as in the Riemannian case, the spin connection $\hat{\Gamma}_{\hat{A}}$ in 5D Weitzenböck space can be expressed as

$$\hat{\Gamma}_{\hat{A}} = \frac{1}{2}\hat{b}_{\hat{i}}{}^{\hat{B}}\hat{\nabla}_{\hat{A}}\hat{b}_{\hat{j}\hat{B}}\hat{\sigma}^{\hat{i}\hat{j}} \tag{2.21}$$

where $\hat{\sigma}^{\hat{i}\hat{j}} = \frac{1}{4}[\gamma^{\hat{i}}, \hat{\gamma}^{\hat{j}}]$, and $\hat{\gamma}^{\hat{i}}$ satisfy $\{\hat{\gamma}^{\hat{i}}, \hat{\gamma}^{\hat{j}}\} = 2\hat{\eta}^{\hat{i}\hat{j}}$. Using (2.17), we can rewrite (2.21) as

$$\hat{\Gamma}_{\hat{A}} = \hat{\tilde{\Gamma}}_{\hat{A}} - \frac{1}{2}\hat{K}_{\hat{i}\hat{j}\hat{A}}\hat{\sigma}^{\hat{i}\hat{j}} \tag{2.22}$$

where $\hat{\Gamma}_{\hat{A}}$ represents the Riemann spin connection in V_5. In Weitzenböck space, the components of a spinor are kept unchanged during parallel translation; therefore, the spin connection $\hat{\Gamma}_{\hat{A}} = 0$. This can also be seen from (2.21) due to $\hat{\nabla}\hat{b} = 0$. Then we have

$$\hat{\tilde{\Gamma}}_{\hat{A}} = \frac{1}{2}\hat{K}_{\hat{i}\hat{j}\hat{A}}\hat{\sigma}^{\hat{i}\hat{j}} \tag{2.23}$$

A straightforward calculation gives

$$\hat{\tilde{\Gamma}}_{A} = \frac{1}{2}K_{ij\mu}\sigma^{ij}\delta^{\mu}_{A} = \tilde{\Gamma}_{\mu}\delta^{\mu}_{A}, \qquad \hat{\tilde{\Gamma}}_{5} = -\frac{1}{2}\kappa F_{ij}\sigma^{ij} \tag{2.24}$$

where $\tilde{\Gamma}_{\mu} = \frac{1}{2}K_{ij\mu}\sigma^{ij}$ is equal to the Riemannian spin connection in 4D spacetime (Hayashi and Shirafuji, 1979), and $F_{ij} = F_{\mu\nu}b^{\mu}_{i}b^{\nu}_{j}$. Hence, from (2.4), the Riemannian spin-covariant derivatives $\hat{\tilde{\nabla}}_{\hat{A}}\Psi = (\hat{e}_{\hat{A}} + \hat{\tilde{\Gamma}}_{\hat{A}})\Psi$ can be written as

$$\begin{aligned} \hat{\tilde{\nabla}}_{A}\Psi &= \delta^{\mu}_{A}(\hat{e}_{\mu} + \hat{\tilde{\Gamma}}_{\mu})\Psi = \delta^{\mu}_{A}(\partial_{\mu} - \kappa A_{\mu}\partial_{\sigma} + \tilde{\Gamma}_{\mu})\Psi \\ \hat{\tilde{\nabla}}_{5}\Psi &= (\hat{e}_{5} + \hat{\tilde{\Gamma}}_{5})\Psi = (\partial_{5} - \frac{1}{2}\kappa F_{\mu\nu}\sigma^{\mu\nu})\Psi \end{aligned} \tag{2.25}$$

3. ACTION AND EQUATION OF DIRAC FIELD

The absolute parallelism of the spinor leads the covariant derivatives of the spinor to coincide with the usual derivatives. So the Dirac Lagrangian density $\hat{L}_D$ in the HLB is given by

$$\hat{L}_D = \frac{1}{2}i\hat{b}_{\hat{k}}{}^{\hat{A}}[\overline{\Psi}\hat{\gamma}^{\hat{k}}\hat{e}_{\hat{A}}\Psi - (\hat{e}_{\hat{A}}\overline{\Psi})\hat{\gamma}^{\hat{k}}\Psi] - m\overline{\Psi}\Psi \tag{3.1}$$

It can further be rewriten as

$$\begin{aligned} \hat{L}_D = {} & \frac{1}{2}i\hat{b}_{k}{}^{\mu}[\overline{\Psi}\gamma^{k}\partial_{\mu}\Psi - (\partial_{\mu}\overline{\Psi})\gamma_{k}\Psi] - m\overline{\Psi}\Psi \\ & + \frac{1}{2}i\hat{b}_{k}{}^{\mu}[\kappa A_{\mu}(\partial_{5}\overline{\Psi})\gamma^{k}\Psi - \overline{\Psi}\gamma^{k}\kappa A_{\mu}\partial_{5}\Psi] \end{aligned}$$

$$+\tfrac{1}{2} i b_5{}^5[\overline{\Psi}\hat{\gamma}^5 \partial_5 \Psi - (\partial_5 \overline{\Psi})\hat{\gamma}^5 \Psi] \tag{3.2}$$

Because the fifth coordinate is usually assumed to be periodic, we choose, as in 5D Riemann space, the Dirac spinor in the form $\Psi(x^\mu, x^5) = L^{-1/2}\exp(ix^5/r)\psi(x^\mu)$ with $0 < x^5 \leq 2\pi r = L$ to guarantee that the 5D theory is covariant with respect to $U(1)$ (Macias and Dehnen, 1991). Thus (3.2) may be expressed in terms of ψ as

$$\begin{aligned}\hat{L}_D = L^{-1}\{&\tfrac{1}{2} i b_k{}^\mu[\overline{\psi}\partial_\mu\psi - (\partial_\mu\overline{\psi})\gamma^k\psi] - m\overline{\psi}\psi \\ &- i b_i{}^\mu \overline{\psi} i e A_\mu \gamma^k \psi - r^{-1} b_5{}^5 \overline{\psi}\hat{\gamma}^5\psi\}\end{aligned} \tag{3.3}$$

where $e = \kappa/r$ is the electronic charge. Rewriting the Dirac Lagrangian density $\hat{L}_D$ by use of the Riemannian covariant derivative $\tilde{\hat{\nabla}} = \partial_\mu - ieA_\mu + \tilde{\Gamma}_\mu$ so as to compare with the cases of Riemann space and Riemann–Cartan space, we get

$$\begin{aligned}\hat{L}_D = L^{-1}\{&\tfrac{1}{2} i b_k{}^\mu[\overline{\psi}\gamma^k \tilde{\hat{\nabla}}_\mu \psi - (\tilde{\hat{\nabla}}_\mu \overline{\psi})\gamma^k\psi] - \tfrac{3}{4} a_k \overline{\psi}\gamma^5\gamma^k\psi \\ &- r^{-1}\overline{\psi}\hat{\gamma}^5\psi - m\overline{\psi}\psi\}\end{aligned} \tag{3.4}$$

where a^μ is the axial vector part of the torsion tensor (Hayashi and Shirafuji, 1979),

$$a^\mu = b_k{}^\mu a^k = \tfrac{1}{6}\epsilon^{\mu\nu\rho\sigma} Q_{\nu\rho\sigma} \tag{3.5}$$

The 5D Dirac action then is

$$\hat{I}_D = \int d^5x\, \hat{b}\hat{L}_D \tag{3.6}$$

where

$$\hat{b} = \det(\hat{b}_i{}^{\hat{A}}) = b = \det(b_i{}^\mu) \tag{3.7}$$

Substituting (3.4) into (3.6), we obtain

$$\begin{aligned}\hat{I}_D = \int d^4x\, b\bigg\{&\frac{1}{2} i b_k{}^\mu[\overline{\psi}\gamma^k\tilde{\hat{\nabla}}_\mu\psi - (\tilde{\hat{\nabla}}_\mu\overline{\psi})\gamma^k\psi] - \frac{3}{4} a_k\overline{\psi}\gamma^5\gamma^k\psi \\ &- m\overline{\psi}\psi - r^{-1}\overline{\psi}\hat{\gamma}^5\psi\}\end{aligned} \tag{3.8}$$

Varying $\hat{I}_D$ with respect to $\overline{\psi}$, we get the Dirac equation of Kaluza–Klein theory in 5D Weitzenböck space:

$$i b_k{}^\mu \gamma^k \tilde{\hat{\nabla}}_\mu \psi - \tfrac{3}{4} a_k \gamma^5\gamma^k\psi - r^{-1}\hat{\gamma}^5\psi - m\psi = 0 \tag{3.9}$$

where the term $\frac{3}{4} a_k\gamma^5\gamma^k\psi$, which occurs in the Dirac equation in 4D Weitzenböck space, is the contact interaction term induced by torsion, while $r^{-1}\hat{\gamma}^5\psi$ represents the additional mass term caused by five-dimensional space.

4. LAGRANGIAN AND FIELD EQUATION OF GRAVITY

The torsion tensor can be decomposed into three irreducible parts under the group of Lorentz transformation Λ_5 (Lee and Wu, 1992)

$$\hat{t}_{\hat{\lambda}\hat{\mu}\hat{\nu}} = \frac{1}{2}(\hat{Q}_{\hat{\lambda}\hat{\mu}\hat{\nu}} + \hat{Q}_{\hat{\mu}\hat{\lambda}\hat{\nu}}) + \frac{1}{8}(\hat{g}_{\hat{\nu}\hat{\lambda}}\,\hat{v}_{\hat{\mu}} + \hat{g}_{\hat{\nu}\hat{\mu}}\hat{v}_{\hat{\lambda}}) - \frac{1}{4}\hat{g}_{\hat{\lambda}\hat{\mu}}\,\hat{v}_{\hat{\nu}} \tag{4.1}$$

$$\hat{v}_{\hat{\mu}} = \hat{Q}^{\hat{\lambda}}{}_{\hat{\lambda}\hat{\mu}} \tag{4.2}$$

$$\hat{a}_{\hat{\rho}\hat{\sigma}} = \frac{1}{3!}\epsilon_{\hat{\lambda}\hat{\nu}\hat{\mu}\hat{\rho}\hat{\sigma}}\,\hat{Q}^{\hat{\lambda}\hat{\mu}\hat{\nu}} \tag{4.3}$$

From the three parts, we can construct the gravitational Lagrangian density

$$\hat{L}_G = \frac{1}{16\pi GL}[\hat{a}_1(\hat{t}^{\hat{\lambda}\hat{\mu}\hat{\nu}}\hat{t}_{\hat{\lambda}\hat{\mu}\hat{\nu}}) + \hat{a}_2(\hat{v}_{\hat{\mu}}\hat{v}^{\hat{\mu}}) + \hat{a}_3(\hat{a}^{\hat{\rho}\hat{\sigma}}\hat{a}_{\hat{\rho}\hat{\sigma}})] \tag{4.4}$$

which can be further reduced, by use of (2.16), to

$$\hat{L}_G = \frac{1}{16\pi GL}\left\{\left[\hat{a}_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu}) + \left(\frac{1}{8}\hat{a}_1 + \hat{a}_2\right)(v^{\mu}v_{\mu}) - 2\hat{a}_3(a^{\rho}a_{\rho})\right] - \left(\frac{\hat{a}_1}{2} + \frac{\hat{a}_3}{9}\right)\kappa^2 F^{\mu\nu}F_{\mu\nu}\right\} \tag{4.5}$$

where $t_{\lambda\mu\nu}$, v_{μ}, and a_{μ} are three irreducible parts under the group of Lorentz transformation Λ_4 (Hayashi and Shirafuji, 1979). The gravitational action is

$$\hat{I}_G = \int d^5x\,\sqrt{\hat{g}}\hat{L}_G \tag{4.6}$$

By the choice $\kappa^2 = 16\pi G$ and (Lee and Wu, 1992)

$$\tfrac{1}{2}\hat{a}_1 + \tfrac{1}{9}\hat{a}_3 = -\tfrac{1}{4} \tag{4.7}$$

the 5D action $\hat{I}_G$ can be reduced to the 4D action and an electromagnetic part

$$\hat{I}_G = \int d^4x\,\sqrt{-g}\,L_G - \frac{1}{4}\int d^4x\,\sqrt{-g}\,F^{\mu\nu}F_{\mu\nu} \tag{4.8}$$

where

$$L_G = \frac{1}{2k}[a_1(t^{\lambda\mu\nu}t_{\lambda\mu\nu}) + a_2(v^{\mu}v_{\mu}) + a_3(a^{\rho}a_{\rho})] \tag{4.9}$$

$k = 8\pi G$ and $a_1 = \hat{a}_1$, $a_2 = \frac{1}{8}\hat{a}_1 + \hat{a}_2$, $a_3 = -2\hat{a}_3$. The gravitational action $\hat{I}_G$ can be writen in another form (Hayashi and Shirafuji, 1979):

$$\hat{I}_G = \int d^4x \sqrt{-g}\, \frac{1}{2k} [\tilde{R} + c_1(t^{\lambda\mu\nu} t_{\lambda\mu\nu}) + c_2(v_\mu v^\mu) + c_3(a_\mu a^\mu)]$$

$$- \frac{1}{4} \int d^4x \sqrt{-g} F^{\mu\nu} F_{\mu\nu} \tag{4.10}$$

where

$$c_1 = a_1 + \frac{2}{3} = \frac{c_3}{9}, \qquad c_2 = a_2 - \frac{2}{3}, \qquad c_3 = a_3 + \frac{3}{2} \tag{4.11}$$

$$\tilde{R} = g^{\mu\nu} \tilde{R}^\lambda{}_{\mu\lambda\nu} \tag{4.12}$$

$$\tilde{R}^\rho{}_{\sigma\mu\nu} = \partial_\mu \tilde{\Gamma}^\rho_{\sigma\nu} - \partial_\nu \tilde{\Gamma}^\rho_{\sigma\mu} + \tilde{\Gamma}^\rho_{\lambda\mu} \tilde{\Gamma}^\lambda_{\sigma\nu} - \tilde{\Gamma}^\rho_{\lambda\nu} \tilde{\Gamma}^\lambda_{\sigma\mu} \tag{4.13}$$

The total action should be

$$\hat{I} = \hat{I}_G + \hat{I}_D \tag{4.14}$$

Varying $\hat{I}$ with respect to b^k_μ, after some algebra, we can split the gravitational field equation into symmetric and antisymmetric parts:

$$\tilde{G}^{ij} + \frac{1}{\sqrt{-g}} \partial_\lambda(\sqrt{-g} F^{(ij)\lambda}) + H^{ij} - \frac{1}{2} \eta^{ij} L' = k T^{(ij)} \tag{4.15}$$

$$2\partial_\lambda(\sqrt{-g} F^{[ij]\lambda}) = k\sqrt{-g} T^{[ij]} \tag{4.16}$$

where

$$\tilde{G}^{ij} = b^i_\mu b^j_\nu \left(\tilde{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \tilde{R} \right) \tag{4.17}$$

$$F^{ij\lambda} = b^i_\mu b^j_\mu F^{\mu\nu\lambda} \tag{4.18}$$

$$F^{\mu\nu\lambda} = \frac{1}{9} c_3 (t^{\mu\nu\lambda} - t^{\mu\lambda\nu}) + c_2 (g^{\mu\nu} v^\lambda - g^{\mu\lambda} v^\nu) - \frac{1}{3} c_3 \epsilon^{\mu\nu\lambda\rho} a_\rho \tag{4.19}$$

$$H^{ij} = b^i_\mu b^j_\nu H^{\mu\nu} = b^i_\mu b^j_\nu \left(T^{\rho\sigma\mu} F_{\rho\sigma}{}^\nu - \frac{1}{2} T^{\nu\rho\sigma} F^\mu{}_{\rho\sigma} \right) \tag{4.20}$$

$$L' = c_1 (t^{\lambda\mu\nu} t_{\lambda\mu\nu}) + c_2 (v^\mu v_\mu) + c_3 (a^\mu a_\mu) \tag{4.21}$$

$$T^{ij} = b^i_\mu b^j_\nu T^{\mu\nu} = \frac{1}{\sqrt{-g}} \eta^{ki} b^j_\nu \frac{\delta\sqrt{-g}(L_D - \frac{1}{4} F^{\mu\nu} F_{\mu\nu})}{\delta b^k_\nu} \tag{4.22}$$

Here T^{ij} is the total energy-momentum tensor of the Dirac and electromagnetic fields, and $T^{(ij)}$ and $T^{[ij]}$ are its symmetric and antisymmetric parts, respectively.

5. THE WEAK-FIELD APPROXIMATION

In order to reveal the nature of the spin–spin interaction induced by torsion in 5D Weitzenböck space, we consider the weak-field approximation (Hayashi and Shirafuji, 1979):

$$b^k{}_\mu(x) = \delta^k{}_\mu + d^k{}_\mu \qquad \left|d^k{}_\mu(x)\right| \ll 1 \tag{5.1}$$

$$b_k{}^\mu(x) = \delta_k{}^\mu + e_k{}^\mu, \qquad \left|e_k{}^\mu(x)\right| \ll 1 \tag{5.2}$$

From (2.1) we get

$$d^\nu{}_\mu + e_\mu{}^\nu = 0 \tag{5.3}$$

Thus, $\{d_{\mu\nu}\}$ can be treated as basic field variables, which can be decomposed into symmetric and antisymmetic parts

$$d_{\mu\nu} = \tfrac{1}{2} h_{\mu\nu} + A_{\mu\nu} \tag{5.4}$$

with $h_{\mu\nu} = h_{\nu\mu}$, $A_{\mu\nu} = -A_{\nu\mu}$. The components of the metric tensor are then written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5.5}$$

By such an approximation, the antisymmetric field makes no contribution to the space-time metric, which implies that it is associated with the intrinsic spin of spin-1/2 fundamental particles. In weak-field case, the axial vector part of the torsion is given by

$$a^\mu = \tfrac{1}{3} e^{\mu\nu\rho\sigma}\, \partial_\nu A_{\rho\sigma} \tag{5.6}$$

The gravitational field equations (4.15) and (4.16) then become

$$\begin{aligned} &-\left(\frac{1}{2} + \frac{3}{2}c_1\right)\partial^\rho\,\partial_\rho \bar{h}_{\mu\nu} + \left(c_1 - \frac{1}{2}c_2 + \frac{1}{2}\right)\partial^\lambda(\partial_\mu \bar{h}_{\nu\lambda} + \partial_\nu \bar{h}_{\mu\lambda}) \\ &\quad - \left(\frac{1}{2} + \frac{1}{2}c_1 - c_2\right)\eta_{\mu\nu}\,\partial_\rho\,\partial_\sigma \bar{h}^{\rho\sigma} + (c_1 + c_2)\partial^\lambda(\partial_\mu A_\nu - \partial_\nu A_\mu) = kT_{(\mu\nu)} \end{aligned} \tag{5.7}$$

$$\begin{aligned} &-\frac{5}{18}c_3\partial^\rho\partial_\rho A_{\mu\nu} + \left(\frac{1}{2}c_2 - \frac{2}{9}c_3\right)\partial^\lambda(\partial_\mu A_{\nu\lambda} - \partial_\nu A_{\mu\lambda}) \\ &\quad - \frac{1}{4}\left(\frac{1}{9}c_3 + c_2\right)\partial^\lambda(\partial_\mu h_{\nu\lambda} - \partial_\nu h_{\mu\lambda}) = 8\pi G T_{[\mu\nu]} \end{aligned} \tag{5.8}$$

where $\bar{h}_{\mu\nu}$ are defined as

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h, \qquad h = \eta^{\mu\nu} h_{\mu\nu} \tag{5.9}$$

Multiplying ∂^ν on both sides of (5.7) and (5.8), we find that, if we choose

$$c_2 = -c_1 = -\tfrac{1}{9}c_3 \tag{5.10}$$

both the symmetric and antisymmetric parts of the angular momenta $T_{\mu\nu}$ satisfy the conservation law

$$\partial_\nu T^{[\mu\nu]} = 0, \qquad \partial_\nu T^{(\mu\nu)} = 0 \tag{5.11}$$

Thus we have the relation (Hayashi and Shirafuji, 1979)

$$2T^{[\mu\nu]} = \partial_\rho S^{\mu\nu\rho} \tag{5.12}$$

where $S^{\mu\nu\rho}$ is the spin tensor of matter. For Dirac particles, it has the totally antisymmetric form

$$S^{\mu\nu\rho} = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}\gamma^5\gamma_\sigma\psi \tag{5.13}$$

It can be checked by use of (5.10) that equations (5.7) and (5.8) are invariant under gauge transformations

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu J_\nu - \partial_\nu J_\mu \tag{5.14}$$

$$A'_{\mu\nu} = A_{\mu\nu} + \partial_\mu H_\nu - \partial_\nu H_\mu \tag{5.15}$$

where J_μ and H_μ are arbitrarily small functions which leave the fields weak. Taking the gauge conditions to be

$$\partial_\nu A^{\mu\nu} = 0, \qquad \partial_\nu \bar{h}^{\mu\nu} = 0 \tag{5.16}$$

we find that the field equations (5.7) and (5.8) become

$$\partial^\rho\partial_\rho \bar{h}_{\mu\nu} = \chi T_{(\mu\nu)} \tag{5.17}$$

$$\partial^\rho\partial_\rho A_{\mu\nu} = -\lambda T_{[\mu\nu]} \tag{5.18}$$

with

$$\chi = \frac{12k}{3 + c_3}, \qquad \lambda = \frac{18k}{5c_3} \tag{5.19}$$

With the help of (5.12) and (5.18), we obtain

$$\partial_\rho A_{\mu\nu} = -\tfrac{1}{2}\lambda S_{\mu\nu\rho} \tag{5.20}$$

Substituting (5.13) and (5.20) into (5.6) gives

$$a_\mu = \tfrac{1}{2}\lambda\bar{\psi}\gamma^5\gamma_\mu\psi \tag{5.21}$$

So the Dirac equation (3.9) in the weak-field approximation can be expressed as

$$ib_k{}^{\mu}\gamma^k\hat{\tilde{\nabla}}_{\mu}\psi - \tfrac{3}{8}\lambda\bar{\psi}\gamma^5\gamma_{\mu}\psi\gamma^5\gamma^{\mu}\psi - r^{-1}\hat{\gamma}^5\psi - m\psi = 0 \tag{5.22}$$

Apart from the additional mass term, the form of the equation is the same as that of the 4D weak-field case, where $c_1 = -c_2$ also must be set in order to guarantee that both $T_{(\mu\nu)}$ and $T_{[\mu\nu]}$ are conserved.

6. SPIN–SPIN INTERACTION

To investigate the spin–spin interaction induced by torsion, as we did in the 4D case (Wu and Zhang, 1998) and 5D EC theory (Zhang and Wu, 1996), we can ignore the connection (Γ_{μ}), the electromagnetic potential (A_{μ}) and the additional mass term ($r^{-1}\gamma^5\psi$) temporarily. Then the Dirac equation (3.9) can be rewritten as

$$i\gamma^{\mu}\partial_{\mu}\psi - \tfrac{3}{4}a_{\mu}\gamma^5\gamma^{\mu}\psi - m\psi = 0 \tag{6.1}$$

assuming that the torsion axial vector a_{μ} can be regarded as the background torsion generated by spin-1/2 particles, which is supposed to be an electron distribution with number density n and spin in the up ($+z$) direction. The background electronic wave function can be taken as $u = u\,(0)\,e^{-ip_0t}$, $u(0) = \sqrt{n}(1\ 0\ 0\ 0)^T$. A test electron put into the background, will suffer an action from the background torsion. Its wave function can be set to $\psi = \psi(0)e^{-iEt}$. Then the Dirac equation (6.1) becomes

$$\gamma^0 m\psi + \tfrac{3}{8}\lambda(\bar{u}\hat{\gamma}_5\gamma^{\mu}u)\hat{\gamma}_5\gamma_{\mu}\psi = E\psi \tag{6.2}$$

In this case, we can take the constant Dirac matirx

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad \hat{\gamma}_5 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{6.3}$$

in the evaluation. Then the eigenenergy for the test particle (in cgs unit) is

$$E = mc^2 \pm \frac{3\lambda\hbar^2}{8c^2}\,n \tag{6.4}$$

Thus we can conclude that in Weitzenböck space the gravitational spin–spin interaction is repulsive for Dirac particles with aligned spins and attractive for opposed spins. If we take the parameter $\lambda = 8\pi G$ ($c_3 = 18/5$), then the interaction in Weitzenböck space is equivalent to that in 4D Riemann–Cartan space (Hehl *et al.*, 1976). But the spin interaction caused by the fifth dimension, unlike in 5D Riemann–Cartan space (Zhang and Wu, 1996), does not exist in 5D Weitzenböck space.

7. DISCUSSION

In order to see explicitly the differences in the Dirac equation in the three kinds of spaces, we list the respective Dirac equation as follows:

- 5D Riemann space (V_5) (Macias and Dehnen, 1991):

$$i\gamma^{\mu}\hat{\tilde{\nabla}}_{\mu}\psi + \frac{1}{2}i\sqrt{16\pi G}F_{\mu\nu}\sigma^{\mu\nu}\gamma^5\psi - r^{-1}\gamma^5\psi - m\psi = 0 \quad (7.1)$$

- 5D Weitzenböck space (A_5):

$$i\gamma^{\mu}\hat{\tilde{\tilde{\nabla}}}_{\mu}\psi - \frac{3}{4}a_{\mu}\gamma^5\gamma^{\mu}\psi - r^{-1}\gamma^5\psi - m\psi = 0 \quad (7.2)$$

- 5D Riemann–Cartan space (U_5) (Zhang and Wu, 1996):

$$\begin{aligned} &i\gamma^{\mu}\hat{\tilde{\nabla}}_{\mu}\psi + \frac{1}{2}i\sqrt{16\pi G}F_{\mu\nu}\sigma^{\mu\nu}\gamma^5\psi - r^{-1}\gamma^5\psi - m\psi \\ &\quad + 3\pi G(\overline{\psi}\gamma_5\gamma^{\mu}\psi)\gamma_5\gamma_{\mu}\psi - \frac{3}{2}\pi G(\overline{\psi}\gamma_{[5\mu\nu]}\psi)\gamma^{[5\,\mu\nu]}\psi = 0 \quad (7.3)\end{aligned}$$

  where $\gamma^{[5\mu\nu]} \equiv \gamma^{[5}\gamma^{\mu}\gamma^{\nu]}$ is antisymmetric with respect to 5, μ, ν.

- 4D Riemann space (V_4):

$$[i\gamma^{\mu}\tilde{\nabla}_{\mu} - m]\psi = 0 \quad (7.4)$$

- 4D Weitzenböck space (A_4):

$$[i\gamma^{\mu}\tilde{\tilde{\nabla}}_{\mu} - \frac{3}{4}a_{\mu}\gamma^5\gamma^{\mu} - m]\psi = 0 \quad (7.5)$$

- 4D Riemann–Cartan space (U_4)

$$[i\gamma^{\mu}\tilde{\tilde{\nabla}}_{\mu} - 3\pi G(\overline{\psi}\gamma_5\gamma^{\mu})\gamma_5\gamma_{\mu} - m]\psi = 0 \quad (7.6)$$

Here $\tilde{\nabla}_{\mu} = \partial_{\mu} + \tilde{\Gamma}_{\mu}$ is the spin-covariant derivative in V_4.

It is easy to see that the Dirac equation in A_5 has the spin–torsion coupling term ($\frac{3}{4}a_{\mu}\gamma_5\gamma^{\mu}\psi$) and lacks the spin–electromagnetism coupling term ($\frac{1}{2}i\sqrt{16\pi G}F_{\mu\nu}\sigma_{\mu\nu}\gamma^5\psi$), while the situation in V_5 is just the reverse. In U_5, both terms exist. In spite of such differences, the fact that there exist in all of three types of Dirac equation the same covariant derivative term $\hat{\tilde{\nabla}}_{\mu}\psi = (\partial_{\mu} - ieA_{\mu} + \tilde{\Gamma}_{\mu})\psi$, which contains both spin connection ($\tilde{\Gamma}_{\mu}$) and electromagnetic potential (A_{μ}), reflects the unification of electromagnetic and gravitational action in 5D theories of gravity.

The 5D Dirac equations all have an additional term $r^{-1}\gamma^5\psi$, which represents the additional mass. There are two interpretations of this term. One of is that (Vladimirov, 1987), making transformation with respect to the spinor $\psi = S\psi' = \exp(\theta\gamma^5)\ \psi$, one can get the total mass $m' = \sqrt{r^{-2} + m^2}$ by a reasonable choice of θ. Although the value $r^{-1} = e/\sqrt{16\pi G} \simeq 2.67 \times 10^{-7}\ g$, is very large, the mass parameter (m) in the

Langrangian can be regarded as imaginary, i.e., $m^2 < 0$. If the value of m^2 is chosen to be near r^{-2}, the total mass $m' = \sqrt{r^{-2} + m^2}$ may coinside with the observed value. Another point of view claims (Macias and Dehnen, 1991) that because of the large value of r^{-1} and the fact that m^2 is positive or zero, the modified mass $m' = \sqrt{r^{-2} + m^2}$ cannot accord with the observed value, and so Kaluza–Klein theory must be rejected. Later this mass problem was solved by introducing a scalar field into the metric in Kaluza–Klein theory (Macias and Dehnen, 1992).

It can be seen from the Dirac equation in the three types of space that the Dirac equation in A_5 has only one extra mass term in contrast to the corresponding 4D Dirac equation, and no additional terms. If simplicity is considered a criterion in formulating a physical theory (i.e., if one thinks that the fewer additional terms that result by generalizing the 4D Dirac equation to the 5D case, the better) one can conclude that in this sense Weitzenböck space is the most suitable one for Kaluza–Klein theory.

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